7 Problem BEAM

7.1 General information

The problem is originally described by a partial differential equation subject to boundary conditions. The semi-discretization in space of this equation leads to a stiff system of n non-linear second order differential equations which is rewritten to first order form, thus providing a stiff system of ordinary differential equations of dimension $2n$. The formulation and data have been taken from [HW96].The INdAM-Bari Test Set group contributed this problem to the test set.The software part of the problem is in the files beam.f available at [MM08].

7.2 Mathematical description of the problem

The problem is of the form

$$z'' = f(t, z, z'), \quad z(0) = z_0 \quad z'(0) = z'_0,$$

with

$$z \in R^n, \quad t \geq 0.$$

The function $f : R^n \to R^n$ is defined by

$$f(t, z, z') = Cv + Du.$$

Here C is the tridiagonal $n \times n$ matrix whose entries are given by

$$\begin{cases} (C)_{11} = 1,\ (C)_{nn} = 3,\ \text{and}\ (C)_{ll} = 2, & l = 2, \ldots, n-1, \\ (C)_{l,l+1} = -\cos(z_l - z_{l+1}), & l = 1, \ldots, n-1, \\ (C)_{l,l-1} = -\cos(z_l - z_{l-1}), & l = 2, \ldots, n, \end{cases}$$

and D is the $n \times n$ bidiagonal matrix whose lower and upper diagonal entries are

$$\begin{cases} (D)_{l,l+1} = -\sin(z_l - z_{l+1}), & l = 1, \ldots, n-1, \\ (D)_{l,l-1} = -\sin(z_l - z_{l-1}), & l = 2, \ldots, n, \end{cases}$$

$v = (v_1, v_2, \ldots, v_n)^{\mathrm{T}}$ is defined by

$$v_l = n^4(z_{l-1} - 2z_l + z_{l+1}) + n^2(\cos(z_l)F_y - \sin(z_l)F_x), \quad l = 1, \ldots, n$$

with $z_0 = -z_1$, $z_{n+1} = z_n$, and u is the column vector of size n solution of the tridiagonal system

$$Cu = g$$

with $g = Dv + (z_1'^2, z_2'^2, \ldots, z_n'^2)^{\mathrm{T}}$.

We write this problem to first order form by defining $w = z'$, yelding a system of $2n$ non-linear differential equations of the form

$$\begin{pmatrix} z \\ w \end{pmatrix}' = \begin{pmatrix} w \\ f(t, z, w) \end{pmatrix}$$

with

$$(z, w)^{\mathrm{T}} \in R^{2n}, \quad t \geq 0.$$

The initial values are

$$\begin{pmatrix} z_0 \\ w_0 \end{pmatrix} = \begin{pmatrix} z_0 \\ z'_0 \end{pmatrix}, \quad \text{where} \quad \begin{cases} z_0 = (0, 0. \ldots, 0)^T \\ z'_0 = (0, 0, \ldots, 0)^T \end{cases}$$

7.3 Origin of the problem

The BEAM problem originates from mechanics and describes the motion of an elastic beam which is supposed inextensible, of length 1 and thin. Moreover, it is assumed that the beam is clamped at one end and a force $F = (F_u, F_v)$ acts at the free end. As coordinate system it is used the angle θ as a function of arc length s and time t. The beam is then described by the equations

$$u(s,t) = \int_0^s \cos\theta(\sigma,t)\, d\sigma, \quad v(s,t) = \int_0^s \sin\theta(\sigma,t)\, d\sigma.$$

In order to obtain the equations of motion for this problem, the Lagrange theory is applied. Let T be the kinetic and U the potential energy defined respectively as follows

$$\begin{aligned} T &= \frac{1}{2}\int_0^1 ((\dot{u}(s,t))^2 + (\dot{v}(s,t))^2)\, ds \\ U &= \frac{1}{2}\int_0^1 ((\theta'(s,t))^2\, ds - F_u(t)u(1,t) - F_v(t)v(1,t). \end{aligned}$$

Here dots and primes denote derivatives with respect to t and s, respectively.
Using the Hamilton principle, the equations of motion are derived. They are

$$\begin{aligned} &\int_0^1 G(s,\sigma)\cos(\theta(s,t)-\theta(\sigma,t))\ddot{\theta}(\sigma,t)\, d\sigma = \\ =\ &\theta''(s,t) + \cos\theta(s,t)F_v(t) - \sin\theta(s,t)F_u(t) \\ -\ &\int_0^1 G(s,\sigma)\sin(\theta(s,t)-\theta(\sigma,t))(\dot{\theta}(\sigma,t))^2\, d\sigma \\ &\theta(0,t) = 0, \qquad \theta'(1,t) = 0 \end{aligned} \tag{II.7.1}$$

where

$$G(s,\sigma) = 1 - \max(s,\sigma)$$

is the Green function for the problem $-w''(s) = g(s), w'(0) = w(1) = 0$.
We discretize the integrals with the midpoint rule:

$$\int_0^1 f(\theta(\sigma,t))\, d\sigma = \frac{1}{n}\sum_{k=1}^n f(\theta_k), \qquad \theta_k = \theta\Big((k-\frac{1}{2})\frac{1}{n}, t\Big), \quad k = 1,\ldots,n.$$

Equations (II.7.1) then become

$$\begin{aligned} &\sum_{k=1}^n a_{\ell k}\ddot{\theta}_k = n^4(\theta_{\ell-1} - 2\theta_\ell + \theta_{\ell+1}) + n^2(\cos\theta_\ell\, F_v - \sin\theta_\ell\, F_u) \\ &-\sum_{k=1}^n g_{\ell k}\sin(\theta_\ell - \theta_k)\dot{\theta}_k^2, \qquad \ell = 1,\ldots,n, \\ &\theta_0 = -\theta_1, \quad \theta_{n+1} = \theta_n, \end{aligned}$$

where

$$a_{\ell k} = g_{\ell k}\cos(\theta_\ell - \theta_k), \qquad g_{\ell k} = n + \frac{1}{2} - \max(\ell, k).$$

TABLE II.7.1: *Run characteristics.*

solver	rtol	atol	h0	mescd	scd	steps	accept	#f	#Jac	#LU	CPU
BIMD	10^{-4}	10^{-4}	10^{-4}	3.53	3.58	60	60	1249	58	59	0.2137
	10^{-7}	10^{-7}	10^{-7}	5.61	6.97	777	777	16197	722	744	2.7308
DDASSL	10^{-4}	10^{-4}		1.83	2.29	29120	28928	30700	243		3.1544
	10^{-7}	10^{-7}		4.63	5.25	51757	51160	56908	768		6.4455
GAMD	10^{-4}	10^{-4}	10^{-4}	3.58	3.59	49	49	1715	49	49	0.2030
	10^{-7}	10^{-7}	10^{-7}	5.49	6.28	459	458	21156	458	459	2.2321
MEBDFI	10^{-4}	10^{-4}	10^{-4}	2.56	1.92	578	559	6447	55	55	0.2284
	10^{-7}	10^{-7}	10^{-7}	5.20	5.26	38693	38645	292234	2054	2054	12.7690
PSIDE-1	10^{-4}	10^{-4}		2.52	2.14	42	36	1096	29	168	0.2303
	10^{-7}	10^{-7}		4.28	5.44	241	208	8006	192	964	1.4806
RADAU	10^{-4}	10^{-4}	10^{-4}	3.57	2.49	62	55	406	43	61	0.2645
	10^{-7}	10^{-7}	10^{-7}	4.24	5.72	71	71	1653	46	60	0.5632
VODE	10^{-4}	10^{-4}		−0.25	1.09	60537	60519	145514	1009	3041	7.3727
	10^{-7}	10^{-7}		4.40	6.48	58132	57793	139394	967	3338	7.8080

In Hairer & Wanner [HW96] the exterior forces are chosen as

$$F_u = -\varphi(t), \quad F_v = \varphi(t), \qquad \varphi(t) = \begin{cases} 1.5\ \sin^2 t, & 0 \le t \le \pi, \\ 0, & \pi \le t, \end{cases}$$

and the initial conditions are taken to be

$$\theta(s,0) = 0, \qquad \dot{\theta}(s,0) = 0.$$

7.4 Numerical solution of the problem

The resulting system of ODEs is integrated for $0 \le t \le 5$, using $n = 40$. Table II.7.1 and Figures II.7.1-II.7.3 present the run characteristics, the behavior of the solution components z_{10}, z_{20}, z_{30} and z_{40} over the interval and the work-precision diagrams, respectively. The computation of the scd values is based on the first 40 components, since they refer to the physically important quantities. The reference solution was computed by RADAU on an Alphaserver DS20E, with a 667 MHz EV67 processor, using double precision `work(1) = uround` $= 1.01 \cdot 10^{-19}$, `rtol = atol = h0` $= 1.1 \cdot 10^{-18}$. For the work-precision diagrams, we used: rtol $= 10^{-(4+m/4)}$, $m = 0, \ldots, 16$; atol = rtol; h0 = rtol for BIMD, GAMD, MEBDFDAE, MEBDFI, RADAU and RADAU5.

With respect to the RADAU and RADAU5 results in Table II.7.1 and Figures II.7.2–II.7.5, we remark that for generality of the test set drivers, we did not use the facility to exploit the special structure of problems. By setting the input parameter `IWORK(9)=40`, and adjusting the Jacobian routine appropriately, RADAU and RADAU5 produce considerably better results.

These results are listed for RADAU in Table II.7.2.

References

[HW96] E. Hairer and G. Wanner. *Solving Ordinary Differential Equations II: Stiff and Differential-algebraic Problems*. Springer-Verlag, second revised edition, 1996.

TABLE II.7.2: *Run characteristics obtained by RADAU with exploited special structure.*

solver	rtol	atol	h0	mescd	scd	steps	accept	#f	#Jac	#LU	CPU
RADAU	10^{-4}	10^{-4}	10^{-4}	3.42	2.49	62	55	406	43	61	0.0869
	10^{-7}	10^{-7}	10^{-7}	4.24	5.72	71	71	1653	46	60	0.1728

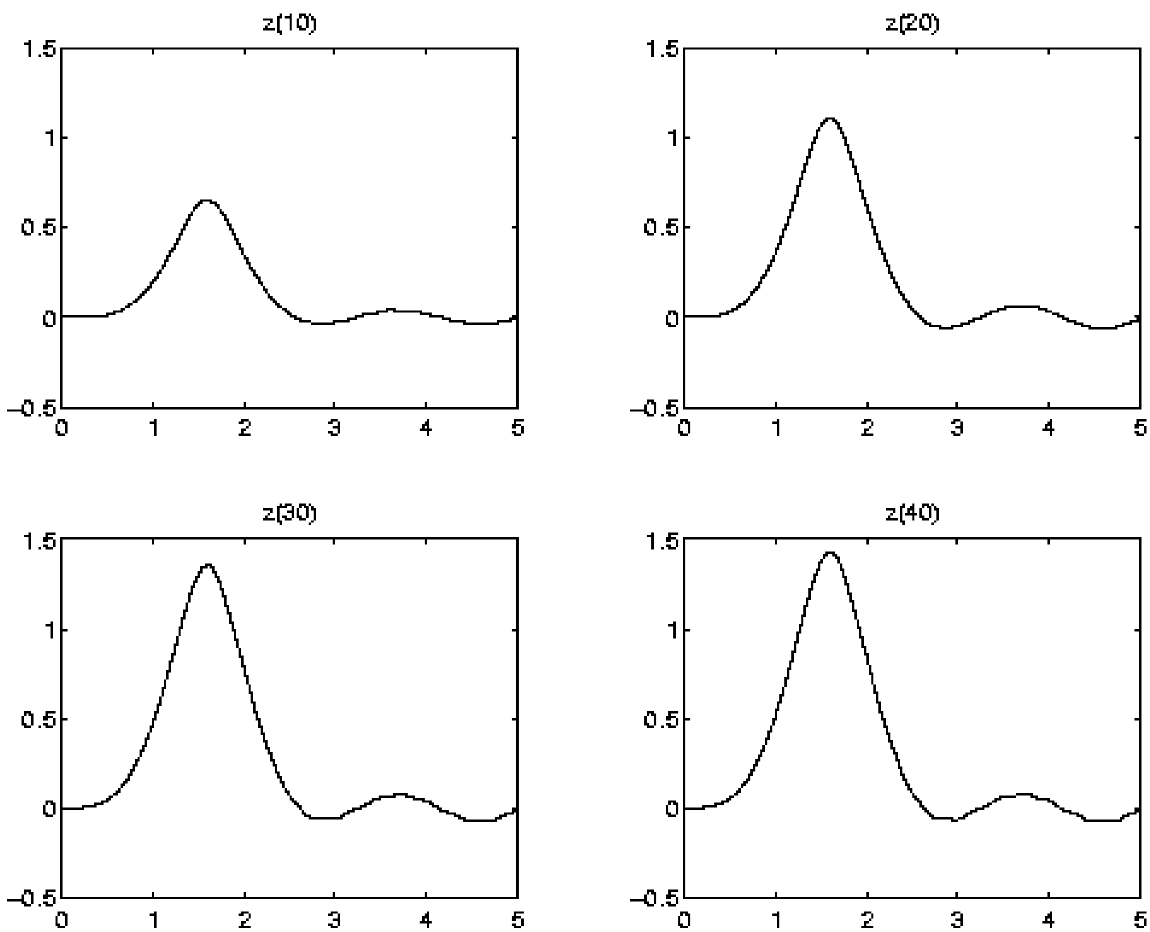

FIGURE II.7.1: Behavior of the solution components z_{10}, z_{20}, z_{30} and z_{40} over the integration interval

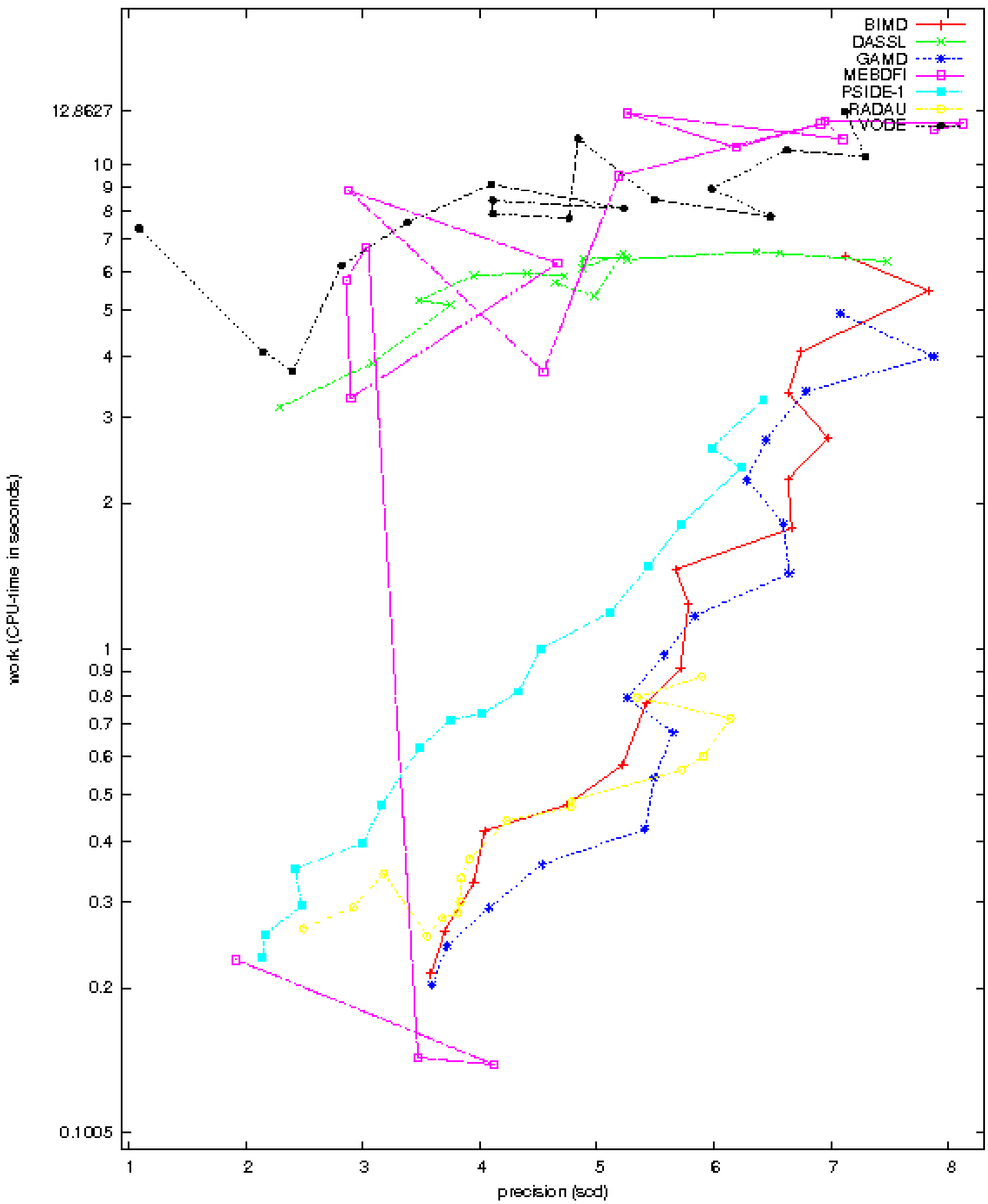

FIGURE II.7.2: *Work-precision diagram (scd versus CPU-time).*

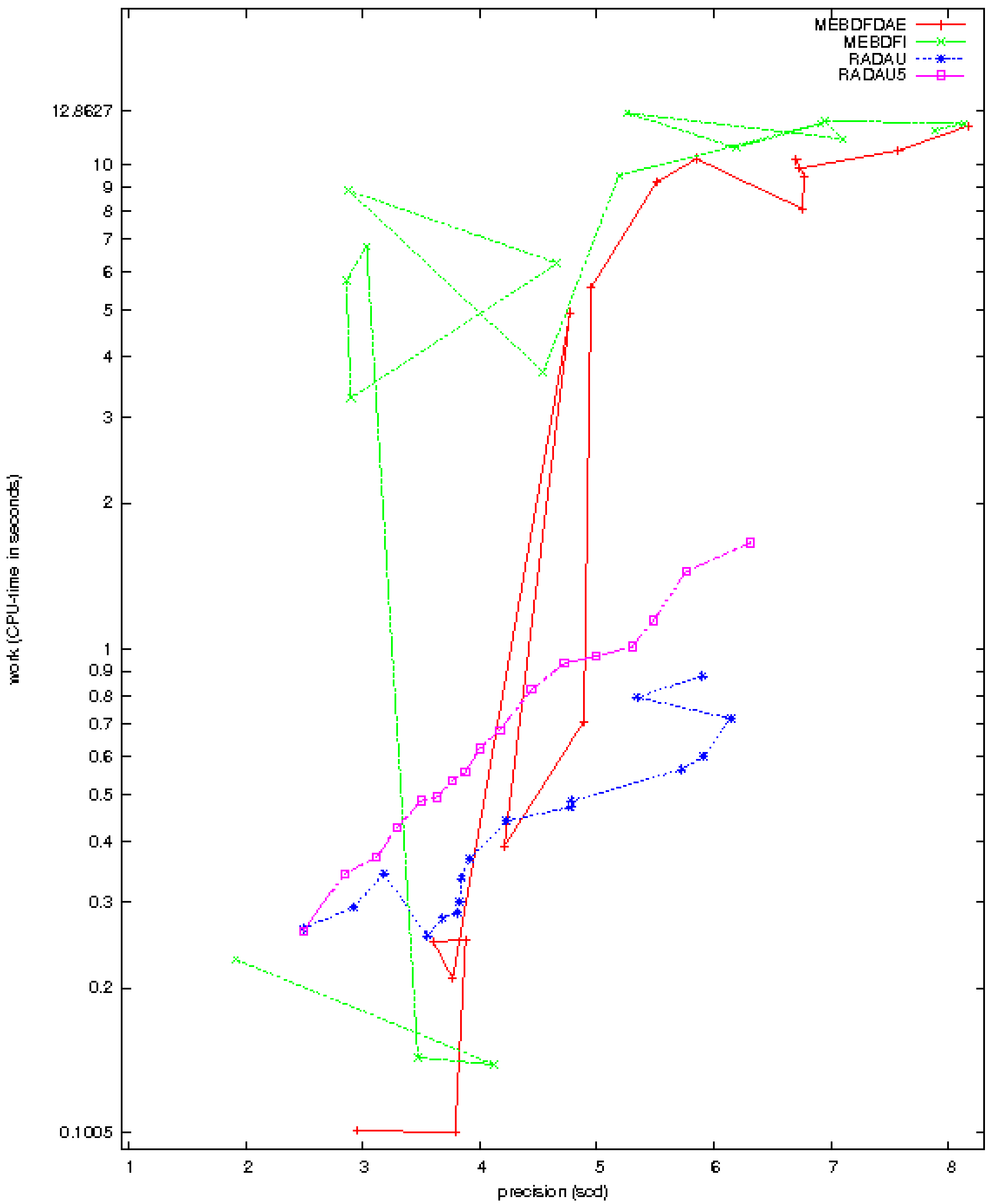

FIGURE II.7.3: *Work-precision diagram (scd versus CPU-time).*

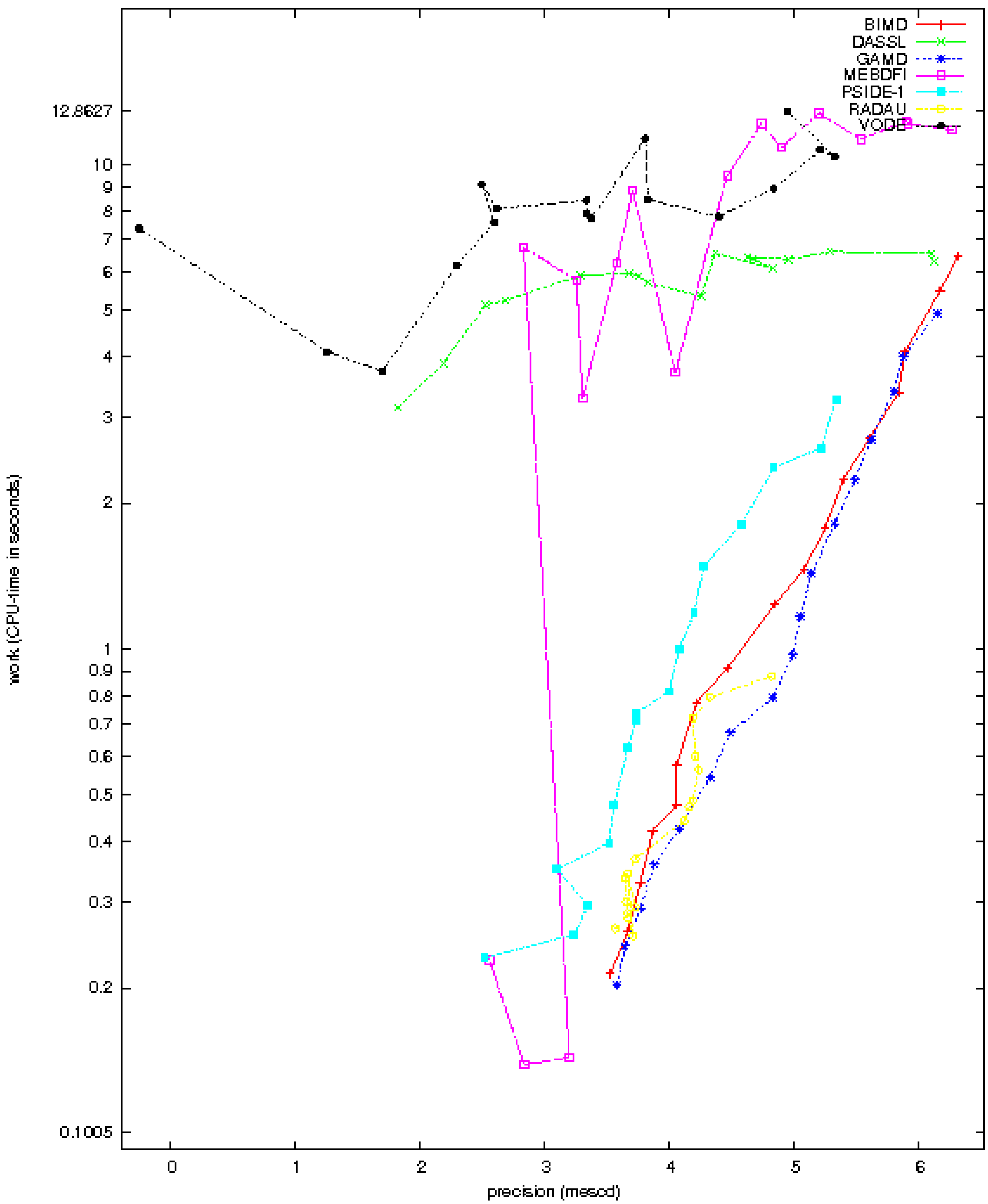

FIGURE II.7.4: *Work-precision diagram(mescd versus CPU-time) .*

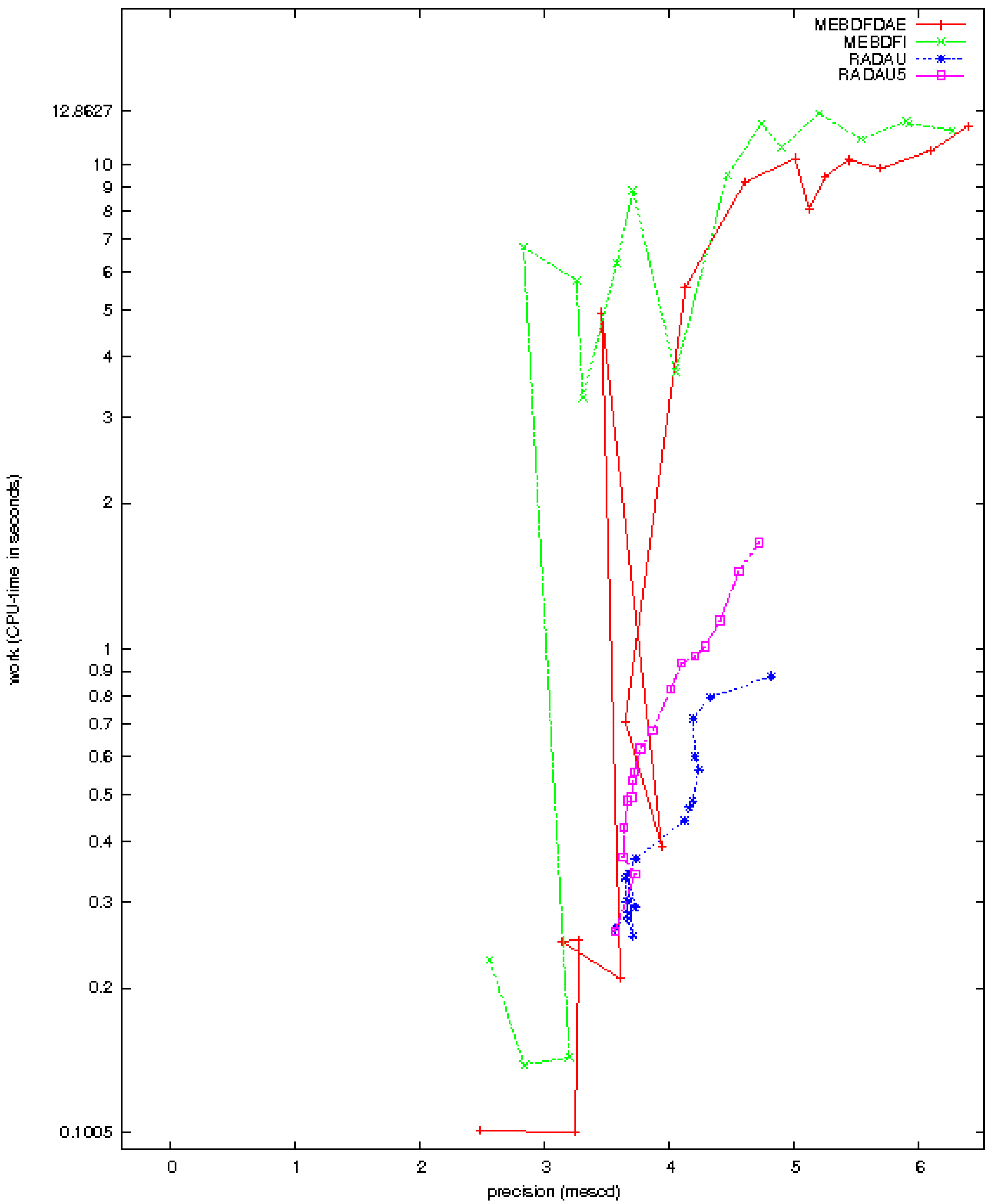

FIGURE II.7.5: *Work-precision diagram (mescd versus CPU-time) .*

[MM08] F. Mazzia and C. Magherini. *Test Set for Initial Value Problem Solvers, release 2.4*. Department of Mathematics, University of Bari and INdAM, Research Unit of Bari, February 2008. Available at http://www.dm.uniba.it/~testset.