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Block-Boundary Value Methods for the solution of Ordinary Differential Equations

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Abstract

Block-Boundary Value Methods are considered and applied to solve initial value problems. Results on their convergence and stability properties are presented and their intermediate position between the class of multistep and Runge-Kutta methods is established.

Key words. Initial Value Problems, Boundary Value Methods, Multistep Methods, Toeplitz Matrices.

AMS(MOS) subject classifications. 65L05, 65L20

1 Introduction

In this paper we are interested in studying some Boundary Value Methods (BVMs) which appear to be suitable for the numerical solution of initial value problems of the form:

$$\begin{cases} \mathbf{y}'(t) &= \mathbf{f}(t,\mathbf{y}) \qquad t \in [a, a+T] \\ \mathbf{y}(a) &= \mathbf{y}_0 \end{cases}, \tag{1}$$

where $\mathbf{y}_0 \in H = \mathbb{R}^m$ and $\mathbf{f}$ is a smooth function $\mathbb{R} \times H \to H$. Under these conditions we denote the solution of (1) with $\hat{\mathbf{y}}(t)$. Concerning the theory on BVMs we will refer to [2, 5, 6, 7, 11, 13].

In its first formulation a BVM simultaneously computes an approximation $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_N)^T$ of the solution of (1) on a grid $a = t_0 < t_1 < \ldots < t_N = a+T$ over the time interval, $t_i = t_{i-1} + h,\ h = T/N$. Such an approximation is obtained as the solution of the algebraic system

$$A^e \mathbf{y}^e - hB^e F(\mathbf{y}^e) = \mathbf{0}, \tag{2}$$

where $F(\mathbf{y}^e) = (\mathbf{f}(t_0, \mathbf{y}_0), \ldots, \mathbf{f}(t_N, \mathbf{y}_N))^T$ and $A^e,\ B^e$ are $N \times (N+1)$ matrices used as linear operators $H^{N+1} \to H^N$. The superscript e appearing on the vector $\mathbf{y}$ and on matrices A^e and B^e, stands for 'extended' and means that the first component of the vector $\mathbf{y}$ is not an unknown. When needed, we will reformulate system (2) in equivalent form, moving all known terms to the right hand side. Performing the following partitions:

$$\begin{aligned} A^e &= [\mathbf{a}_0, A]; \\ B^e &= [\mathbf{b}_0, B]; \\ \mathbf{y}^e &= [\mathbf{y}_0, \mathbf{y}^T]^T; \end{aligned}$$

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this formulation is

$$Ay - hBF(\mathbf{y}) = \mathbf{b}, \tag{3}$$

where $\mathbf{b} = -\mathbf{a}_0\mathbf{y}_0 + h\mathbf{b}_0\mathbf{f}(t_0, \mathbf{y}_0)$ contains information from the preceding step ($\mathbf{a}_0$ and $\mathbf{b}_0$ are again used as linear operators $H \to H^N$) and now the function F is defined on H^N. Equations (2) and (3) may be viewed as a global discrete representation of the IVP (1) and indeed one can estimate the error in the approximation by simply evaluating the distance between $\mathbf{y}$ and the projection of $\hat{\mathbf{y}}(t)$ on H^N, say $\hat{\mathbf{y}} = (\hat{\mathbf{y}}(t_1), \ldots, \hat{\mathbf{y}}(t_N))^T$. This approach is attractive for a number of reasons such as, for example, the possibility of control of the global error and of handling indifferently both initial and boundary value problems [4]. Implementations have been carried out in the case when F is linear together with an effective technique for the stepsize control [2]. Results on the existence, the uniqueness and the stability of the solution of (2) have been derived for the class of monotone problems in which the one-sided Lipshitz condition

$$< \mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y}), \mathbf{x} - \mathbf{y} > \leq \mu ||\mathbf{x} - \mathbf{y}||_2^2, \qquad \forall \mathbf{x}, \mathbf{y} \in H, \ t \in \mathbb{R}$$

is satisfied for some $\mu \in \mathbb{R}$ [11]. It has been observed that no significant extra difficulty (with respect to the linear case) arises when μ is negative (dissipative problems) [13].

When dealing with general problems, some drawbacks emerge. High nonlinearity of the function $\mathbf{f}$ can produce severe obstacles in the management of the system (2). Firstly, until the stepsize h is small enough (even smaller than that required by the error tolerance), the sufficient conditions that guarantee the existence and the uniqueness of the solution of (2) are no longer satisfied. Furthermore, granted that a unique solution exists for a given h, because of the irregularity of $\mathbf{f}$, the iterative method for finding the zero of the nonlinear system (2) could be very sensitive to the initial condition. Due to the fact that the dimension of (2) is inversely proportional to the stepsize h, the computational effort in solving equation (2) becomes relevant when high accuracy is needed. It must be also emphasized that, in the nonlinear case, stepsize control that allows nonuniform grid selection is a difficult task to handle.

These unpleasant aspects all have a common denominator: the dependence of the dimension of the system on the grid-length N. A natural strategy to prevent these awkward situations is to bound N in order to handle systems of relatively small dimensions, say $N \leq \tilde{N}$. In this way a finite number of contiguous blocks

$$W_s = [t^{(s-1)}, t^{(s)}], \qquad s = 1, \ldots, M, \ t^{(0)} = a, \ t^{(M)} = a + T,$$

are defined to cover the entire integration interval. A BVM is applied to integrate $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$ in the block W_s. Consequently M systems of the form

$$A_s\mathbf{y}^{(s)} - h_sB_sF(\mathbf{y}^{(s)}) = \mathbf{b}_s, \qquad s = 1, \ldots, M, \tag{4}$$

of dimensions $N_s \leq \tilde{N}$ have to be solved instead of the larger system represented by equation (3). The s-th system of (4) constructs an approximation to the true solution $\hat{\mathbf{y}}(t)$ in the block W_s by means of the vector $\mathbf{y}^{(s)} = [\mathbf{y}_1^{(s)}, \ldots, \mathbf{y}_{N_s}^{(s)}]^T$, which is defined on the uniform mesh $t^{(s-1)} = t_0^{(s)} < t_1^{(s)} < \ldots < t_{N_s}^{(s)} = t^{(s)}$. The number of unknowns is N_s if we assume that $\mathbf{y}_0^{(s)}$ coincides with $\mathbf{y}_{N_{s-1}}^{(s-1)}$ which has been previously computed as the last component of $\mathbf{y}^{(s-1)}$. The superscript s in (4) suggests the possibility of using, in each block W_s, different dimensions N_s (or equivalently different stepsizes h_s), as well as different BVM formulae. When this approach is adopted it is said that BVMs are applied in a composite technique and they are referred to as composite or block Boundary Value Methods. There are several ways of choosing and managing the blocks W_s and the dimensions N_s. Brugnano and Trigiante showed that global control of the solution in the time domain is still possible by means of an iterative technique that increases the number and the dimension of the blocks in the subregions where the solution needs to be well represented. Their approach leads to relevant results expecially when applied to boundary value problems (see [7]).

The numerical integration which we are interested in proceeds according to the following sequential scheme:

- fix a BVM, h_1 and N_1 (and hence the block $W_1 = [t^{(0)}, t^{(1)}]$) ;
- set $\mathbf{y}_0^{(1)} = \mathbf{y}_0$ and solve the IVP

$$\left\{ \begin{array}{rcl} \mathbf{y}'(t) & = & \mathbf{f}(t, \mathbf{y}) \qquad t \in W_1 \\ \mathbf{y}(a) & = & \mathbf{y}_0 \end{array} \right. ,$$

  by means of the system

$$A_1 \mathbf{y}^{(1)} - h_1 B_1 F(\mathbf{y}^{(1)}) = \mathbf{b}_1,$$

  obtaining $\mathbf{y}^{(1)} = [\mathbf{y}_1^{(1)}, \ldots, \mathbf{y}_{N_1}^{(1)}]^T$;
- set $s = 1$;
- do while $t^{(s)} < a + T$
  - s=s+1;
  - fix a BVM, h_s and N_s (and consequently the block $W_s = [t^{(s-1)}, t^{(s)}]$) ;
  - set $\mathbf{y}_0^{(s)} = \mathbf{y}_{N_{s-1}}^{(s-1)}$ and solve the IVP

$$\left\{ \begin{array}{rcl} \mathbf{y}'(t) & = & \mathbf{f}(t, \mathbf{y}) \qquad t \in W_s \\ \mathbf{y}(t^{(s-1)}) & = & \mathbf{y}_0^{(s)} \end{array} \right. ,$$

    by means of the system

$$A_s \mathbf{y}^{(s)} - h_s B_s F(\mathbf{y}^{(s)}) = \mathbf{b}_s,$$

    and obtain $\mathbf{y}^{(s)} = [\mathbf{y}_1^{(s)}, \ldots, \mathbf{y}_{N_s}^{(s)}]^T$;
- enddo.

If one looks at the family of equations (4) and the way each element is created (as described by the above scheme), one realizes that block-BVMs may be regarded as a subset of the wider class of General Linear Methods (GLMs) as defined by Butcher in [8]. In the next section we formalize the description of the matrices A_s and B_s appearing in (4) and exploit the theory of GLMs to derive results about convergence and order of accuracy. Section 3 is devoted to the study of linear stability properties of block-BVMs while in section 4 we describe a particular family of methods which has been implemented in a code, analyzing its properties in relation to the presented results .

2 Convergence of block-BVMs

As described in the introduction, we fix a family of BVMs and pick elements of this family to perform the integration procedure. In the last section we will focus our attention to a set of methods which can be gathered to form the family (4): at the moment we give an account on how each element of the family is defined. For this reason, to simplify the notation, we omit the subscript (superscript)

s and therefore while examining a block-BVM formula we can directly refer to the systems (2) and (3).

Keeping in mind that now the dimension N is a constant independent of h, formulae (2) and (3) are analogous to those used to find the internal stages of a Runge-Kutta method and indeed it is easily seen that it is possible to reformulate a block-BVM as a particular RK formula. Nevertheless, as will be clear in the following, it would be very restrictive to think of block-BVMs merely as RK methods. As a matter of fact all the existing theory on BVMs may be seen as a generalization of the classical theory of Multistep Methods. We intend to show hereafter and in the rest of the paper how this dual aspect of block-BVMs combines results, properties and techniques separately developed in the fields of Multistep and Runge-Kutta methods. These are, for example, convergence and order of accuracy, the estimation of the error, the change of the steplength, the change of the order and so on. Some of these features are described in detail in [14] where a code based on block-BVMs is presented.

In order to elucidate the link between block-BVMs and Multistep Methods, we need to mention how the elements of the matrices A^e and B^e are generated. The i-th component of the system (2) is actually a $k^{(i)}$-step linear multistep formula with $k_1^{(i)}$ initial and $k_2^{(i)}$ final conditions, $k^{(i)} = k_1^{(i)} + k_2^{(i)}$, which we write as

$$\sum_{j=-k_1^{(i)}}^{k_2^{(i)}} \alpha_{ij} \mathbf{y}_{i+j} = h \sum_{j=-k_1^{(i)}}^{k_2^{(i)}} \beta_{ij} \mathbf{f}_{i+j}. \tag{5}$$

To preserve the compatibility between the number of initial and final conditions and the size $(N+1) \times N$ of the matrices A^e and B^e we must require that $k_1^{(i)} \leq i$ and $k_2^{(i)} \leq N - i$ and consequently $k^{(i)} \leq N$. In practice one can look at the coefficients α_{ij} and βij as the elements of the matrices A^e and B^e with α_{i0} and β_{i0} their diagonal entries on the i-th row.

In short, formula (5) computes $\mathbf{y}_i$, an approximation of $\hat{\mathbf{y}}_i$, by means of approximations to the solution in $k_1^{(i)}$ left points (initial conditions) and $k_2^{(i)}$ right points (final conditions). These extra unknowns $\mathbf{y}_j,\ j = 1, \ldots, k_1^{(i)} - 1, k_1^{(i)} + 1, \ldots, k^{(i)}$ are implicitly determined by the remaining $N-1$ equations of system (2). In particular the last component of (2) coincides with a classical multistep formula since no final conditions are required. If each multistep formula has order p the local truncation error satisfies

$$\boldsymbol{\tau}(h) \equiv A^e \hat{\mathbf{y}}^e - hB^e F(\hat{\mathbf{y}}^e) = h^{p+1} G(\boldsymbol{\xi}), \tag{6}$$

where $G(\boldsymbol{\xi}) = [c_1 \hat{\mathbf{y}}^{(p+1)}(\boldsymbol{\xi}_1), c_2 \hat{\mathbf{y}}^{(p+1)}(\boldsymbol{\xi}_2), \ldots, c_N \hat{\mathbf{y}}^{(p+1)}(\boldsymbol{\xi}_N)]^T$, with c_i the error constant of the i-th formula. Such a simple expression of the truncation error is typical of BVMs and rather difficult to obtain when directly dealing with RK formulae. Formula (5) is also well defined by the polynomial pair (ρ_i, σ_i)

$$\rho_i(\xi) = \xi^{k_1^{(i)}} \sum_{j=-k_1^{(i)}}^{k_2^{(i)}} \alpha_{ij} \xi^j, \qquad \sigma_i(\xi) = \xi^{k_1^{(i)}} \sum_{j=-k_1^{(i)}}^{k_2^{(i)}} \beta_{ij} \xi^j, \tag{7}$$

and hence a block-BVM formula such as (2) is identified by the finite sequence

$$(\rho_i, \sigma_i)_{i=1,\ldots,N}. \tag{8}$$

Historically, such a way of assembling multistep formulae is not completely new: an example may be found in [21] where the authors fixed particular formulae, called block-Implicit Methods, analyzing their convergence and stability properties and pointing out some advantages to Runge-Kutta schemes. However, as will be clear in the sequel, a peculiar element which makes block-BVMs different from any standard one-step method is that once a formula has been fixed, the dimension of the matrices A^e and B^e may be conveniently changed and need not be constant during the integration procedure. Of special interest therefore is the particular pair, say $(\rho_b, \sigma_b) \equiv (\rho_l, \sigma_l)$ with $1 \leq l \leq N$, the so called basic method. The length of the sequence (8) may be increased to a value $N_1 > N$, adding $r = N_1 - N$ basic methods. This corresponds to enlarging the dimension of the system (2) and allows one to cover a wider block or to use in the same block a smaller h in order to reach a higher precision in the solution. Therefore, it being possible that the basic method may appear in the sequence more than one time, we may identify (8) simply by listing the pairs corresponding to different formulae. More precisely, if the basic method is repeated r times, we call initial methods the pairs preceeding the basic method, namely $(\rho_i, \sigma_i)_{i=1,\ldots,l-1}$, and final methods the pairs that follow the basic method, that is $(\rho_i, \sigma_i)_{i=l+r,\ldots,N}$. Hence we can identify a block-BVM by listing either the element of the matrices A^e and B^e or the initial, basic and final methods. The pair (ρ_b, σ_b) should be suitably chosen, because if a block-BVM has a large number of basic methods, it inherits properties satisfied by the basic method itself. We shall see later that the same may hold true when few (possibly one) basic methods are used, under a convenient choice of the initial and final methods.

We derive convergence properties of block-BVMs using the theory of General Linear Methods (GLMs). These are implicit one-step methods introduced by Butcher and are defined by the formula

$$\mathbf{y}^{(n)} = \mathcal{A}\mathbf{y}^{(n-1)} + h\mathcal{B}F(\mathbf{y}^{(n)}) + h\mathcal{C}F(\mathbf{y}^{(n-1)}) \tag{9}$$

where the vector $\mathbf{y}^{(n)} = [\mathbf{y}_1^{(n)}, \ldots, \mathbf{y}_N^{(n)}]^T$ contains approximations of the solution at particular points $t^{(n-1)} + \nu_i h,\ \nu_i > 0$ and $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are $N \times N$ matrices used again as linear operators. As concerns the theory on GLMs we refer to the results presented in [8]. A block-BVM is easily put in the form (9) by means of the following identities:

$$\mathcal{A} = -A^{-1}[\mathbf{0}, \ldots, \mathbf{0}, \mathbf{a}_0],$$

$$\mathcal{B} = A^{-1}B,$$

$$\mathcal{C} = A^{-1}[\mathbf{0}, \ldots, \mathbf{0}, \mathbf{b}_0].$$

Nonetheless it is more convenient to mantain for a block-BVM the expression (2), since in this way it is easier to link properties of the pairs (ρ_i, σ_i) to those needed by a GLM to gain convergence and order of accuracy.

The concept of consistency corresponds to requiring a faithful representation of the solution when a GLM is applied to the test problems $y' = 0$ and $y' = 1$. In our case, defining the vectors $\mathbf{u} = [1, \ldots, 1]^T$ and $\mathbf{v} = [0, 1, \ldots, N-1, N]^T$, the standard definition of consistency for GLMs assumes the following form:

Definition 2.1 *A block-BVM is preconsistent if*

$$A^e\mathbf{u} = \mathbf{0}, \tag{10}$$

and consistent if in addition

$$A^e\mathbf{v} - B^e\mathbf{u} = \mathbf{0}, \tag{11}$$

The theorem reported below shows how one can construct block-BVMs by simply working as usual on each pair (ρ_i, σ_i).

THEOREM 2.2 *A block-BVM* (A^e, B^e) *is consistent iff each multistep formula* (ρ_i, σ_i) *is consistent.*

Proof. Formula (10) is equivalent to

$$\rho_i(1) = 0 \qquad \forall i = 1, \ldots, N. \tag{12}$$

The i-th component of (11) is

$$\sum_{j=-k_1^{(i)}}^{k_2^{(i)}} \alpha_{ij}(i+j) - h \sum_{j=-k_1^{(i)}}^{k_2^{(i)}} \beta_{ij} = 0$$

which because of (12) may be written as

$$\rho_i'(1) = \sigma_i(1).$$

□

To give a convergence result we must investigate the stability of the matrix $\mathcal{A}$ which, by definition, means that $\mathbf{y}^{(n)} = \mathcal{A}^n \mathbf{y}^{(0)}$ is bounded as n tends to infinity for any $\mathbf{y}^{(0)}$. A sufficient condition for stability is that all eigenvalues of $\mathcal{A}$ have modulus less than or equal to one and those of unitary modulus are simple.

THEOREM 2.3 *A preconsistent block-BVM* (A^e, B^e) *is stable.*

Proof. As already observed the matrix $\mathcal{A}$ for a block-BVM may be expressed as

$$\mathcal{A} = -[\mathbf{0}, \ldots, \mathbf{0}, A^{-1}\mathbf{a}_0]$$

and hence $\mathcal{A}$ has $N-1$ eigenvalues at zero. We prove that the remaining eigenvalue is 1 and corresponds to the eigenvector with 1 in each component. From preconsistency we have $A^e\mathbf{u} = \mathbf{0}$. With the partition $\mathbf{u} = [1, \mathbf{u}^{(1)T}]^T$, it follows that

$$A^e\mathbf{u} = \mathbf{a}_0 + A\mathbf{u}^{(1)} = ([\mathbf{0}, \ldots, \mathbf{0}, \mathbf{a}_0] + A)\,\mathbf{u}^{(1)} = -A^{-1}\,(\mathcal{A} - I)\,\mathbf{u}^{(1)}.$$

Consequently (10) becomes

$$\mathcal{A}\mathbf{u}^{(1)} = \mathbf{u}^{(1)},$$

which completes the proof. □

A more intimate inspection leads to the following equivalent expression for the matrix $\mathcal{A}$ belonging to a preconsistent block-BVM:

$$\mathcal{A} = [\mathbf{0}, \ldots, \mathbf{0}, \mathbf{u}^{(1)}].$$

A property that will be exploited in the next theorem is that that $\mathcal{A}$ is idempotent and consequently any power of $\mathcal{A}$ is uniformly bounded; in particular one has $||\mathcal{A}^n||_\infty = 1$ whatever the power n and the dimension N.

From the theory of GLMs we conclude that for a block-BVM convergence is equivalent to consistency. Bounds on the global error may be easily derived considering the difference between (2) and (6):

$$A^e\left(\hat{\mathbf{y}}^{e(n)}-\mathbf{y}^{e(n)}\right)-hB^e\left(F(\hat{\mathbf{y}}^{e(n)})-F(\mathbf{y}^{e(n)})\right)=\boldsymbol{\tau}^{(n)}(h). \tag{13}$$

We introduce the error vector $\mathbf{e}^{(n)}$, defining each component as

$$\mathbf{e}_i^{(n)}=\hat{\mathbf{y}}_i^{(n)}-\mathbf{y}_i^{(n)}, \qquad i=1,\dots,N.$$

Hence equation (13) becomes

$$\begin{cases}\mathbf{e}^{(n)}=\mathcal{A}\mathbf{e}^{(n-1)}+\mathbf{w}^{(n)},\\ \mathbf{e}^{(0)}=\mathbf{0},\end{cases}, \tag{14}$$

where

$$\mathbf{w}^{(n)}=h\mathcal{B}\left(F(\hat{\mathbf{y}}^{(n)})-F(\mathbf{y}^{(n)})\right)+h\mathcal{C}\left(F(\hat{\mathbf{y}}^{(n-1)})-F(\mathbf{y}^{(n-1)})\right)+A^{-1}\boldsymbol{\tau}^{(n)}(h). \tag{15}$$

THEOREM 2.4 *If sufficient smoothness conditions on* $\mathbf{f}$ *are satisfied and the block-BVM* (A^e,B^e) *has order* p*, then there exist positive constants* h_0 *and* S *such that the global truncation error is bounded by*

$$||\mathbf{e}^{(n)}||\le Sh^p, \tag{16}$$

whenever $h<h_0$ *and* $t_n\in[a,a+T]$.

Proof. Working with the infinity norm, let L be the Lipshitz constant of $\mathbf{f}$ and $h<h_0$ where h_0 satisfies

$$L||\mathcal{B}||h_0<1. \tag{17}$$

From (15) we obtain

$$||\mathbf{w}^{(n)}||\le\beta||\mathbf{e}^{(n)}||+\gamma||\mathbf{e}^{(n-1)}||+\delta,$$

where

$$\beta=h||\mathcal{B}||L;$$

$$\gamma=h||\mathcal{C}||L;$$

$$\delta=\max_{1\le i\le N}\{c_i\}\max_{a\le t\le a+T}||\hat{\mathbf{y}}^{(p+1)}(t)||\;||A^{-1}||\;h^{p+1}\equiv S_1h^{p+1}.$$

Considering that $||\mathcal{A}^n||=1$, from this inequality and (14) we deduce (see [8], Lemma 425A, page 362):

$$||\mathbf{e}^{(n)}||\le\frac{\delta}{\beta+\gamma}\left[\left(\frac{1+\gamma}{1-\beta}\right)^n-1\right]. \tag{18}$$

Using the upper bounds

$$\left(\tfrac{1+\gamma}{1-\beta}\right)^n \le e^{n\left(\frac{\gamma+\beta}{1-\beta}\right)},$$

$$nh \le T,$$

and substituting in (18) the values of β, γ, δ, the assertion follows with

$$S = \frac{S_1}{L(||\mathcal{B}|| + ||\mathcal{C}||)}\left[e^{LT\left(\frac{||\mathcal{C}||+||\mathcal{B}||}{1-h_0||\mathcal{B}||L}\right)} - 1\right]. \tag{19}$$

□

Contrary to what happens to a linear multistep formula (ρ, σ), zero-stability is not a necessary condition to obtain convergence. Obviously this is natural in this context, since we are dealing with one-step methods, for which the root condition is a direct consequence of the definition. Nonetheless, expecially when the dimension N is variable and $\tilde{N}$ is large, zero-stability of the basic method (ρ_b, σ_b) should be a desirable feature to be considered when defining a block-BVM. In this respect, we recall that zero-stability of a pair (ρ, σ) defined in (7) and used with k_1 initial and k_2 final conditions, is equivalent to the property of the polynomial ρ to be of type (m_1, m_2, k_2), $m_1 + m_2 = k_1$, namely to possess k_1 roots on the unit circle of which m_1 are inside and m_2 are on its boundary but simple, and the remaining are outside the unit circle. This corresponds to the natural generalization to BVMs of the concept of zero-stability of a linear multistep formula used with only initial conditions and indeed one obtains the classical root condition by imposing $k_2 = 0$. If (ρ_b, σ_b) is zero-stable we then say that the corresponding block-BVM is zero-stable.

Some of the most important block-BVMs, such as those belonging to the Adams-type methods, are characterized by a sequence (8) in which all terms $\rho_i/\xi^{k_1^{(i)}}$ coincide. This leads to a Toeplitz structure for the matrix A. The norm of the inverse of such matrices was analyzed in [1] and put in relation to their dimensions. The result in which we are interested is summarized below. Here we consider a banded Toeplitz matrix $\mathcal{T}_N$ of size N with k_2 upper-diagonals and k_1 lower-diagonals

$$\mathcal{T}_N = \begin{pmatrix} t_0 & t_1 & \cdot & t_{k_2} & & & & & & \\ t_{-1} & \cdot & \cdot & \cdot & t_{k_2} & & & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\ t_{-k_1} & \cdot & t_{-1} & t_0 & t_1 & \cdot & t_{k_2} & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & t_{-k_1} & \cdot & t_{-1} & t_0 & t_1 & \cdot & t_{k_2} \\ & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & \cdot & \cdot & \cdot & \cdot & t_1 \\ & & & & & & t_{-k_1} & \cdot & t_{-1} & t_0 \end{pmatrix}_{N\times N}$$

and its associated polynomial

$$p(z) = \sum_{i=-k_1}^{k_2} t_i z^{k_1+i}.$$

THEOREM 2.5 *Consider the family of Toeplitz matrices $\mathcal{T}_N$ with associated polynomial $p(z)$ and with the constraint $p(1) = 0$. Set*

$$\mu(N,p) = ||\mathcal{T}_N^{-1}||$$

where $||\cdot||$ denotes either $||\cdot||_1$ or $||\cdot||_\infty$. Then (a) $\Rightarrow$ (b), where *(a) $p(z)$ is of type (m_1, m_2, k_2), $m_1 + m_2 = k_1$ or of type (k_1, m_1, m_2), $m_1 + m_2 = k_2$ with simple roots on the boundary of the unit circle.*

(b) $\mu(N, p) = O(N)$;

When condition (a) is not satisfied, $\mu(N, p)$ grows with an order greater than or equal to one. In particular, when $p(z)$ has more than k_1 roots inside the unit circle or more than k_2 roots outside, then $\mu(N, p)$ grows exponentially with respect to N.

Proof. See [1] □

Looking now at formula (17) and the condition number S as defined in (19) and appearing in the bound of $\mathbf{e}^{(n)}$, it is reasonable to require that the quantities $||\mathcal{B}||, ||\mathcal{C}||$ and therefore $||A^{-1}||$ do not grow rapidly with respect to the dimension N. The above result shows that, working in the subclass of block-BVMs that generate a Toeplitz matrix A, if one wants to bound as strong as possible the quantities $||A^{-1}||$ with respect to N and under the order condition $\rho(1) = 0$, the best that can be done is to obtain the weak well conditioning result expressed by the statement (b) in the theorem, by choosing zero-stable block-BVMs.

In the general case, a block BVM produces a family of quasi-Toeplitz matrices as defined and studied in [3]. It may be observed that zero-stability still plays an analogous role. The following example is in this direction.

EXAMPLE Consider the two-step method (ρ_b, σ_b)

$$\rho_b(\xi) = \xi^2 - (1+\alpha)\xi + \alpha, \qquad \sigma_b(\xi) = \xi^2 - \alpha\xi. \tag{20}$$

One can verify that for $\alpha \neq 1$ the method has order one and the roots of $\rho_b(\xi)$ are $\xi_1 = 1$ and $\xi_2 = \alpha$. We use it as the basic method inside two different block-BVMs, say (A_1^e, B_1^e) and (A_2^e, B_2^e). The former is defined assuming that (20) is performed with $k_1 = 2$ and coupled with the initial method (theta-method)

$$\rho(\xi) = \xi - 1, \qquad \sigma(\xi) = (1-\theta)\xi + \theta, \tag{21}$$

used with $\theta = .25$.

The latter has still (20) as basic method but now performed with $k_1 = 1$ (and therefore $k_2 = 1$) and uses (21) as final method. For instance, when used with minimal dimension, these methods assume the form:

$$A_1^e = \begin{pmatrix} -1 & 1 & 0 \\ \alpha & -(1+\alpha) & 1 \end{pmatrix}, \qquad B_1^e = \begin{pmatrix} \theta & (1-\theta) & 0 \\ 0 & -\alpha & 1 \end{pmatrix}$$

and

$$A_2^e = \begin{pmatrix} \alpha & -(1+\alpha) & 1 \\ 0 & -1 & 1 \end{pmatrix}, \qquad B_2^e = \begin{pmatrix} 0 & -\alpha & 1 \\ 0 & \theta & (1-\theta) \end{pmatrix}.$$

According to the definition stated above, we see that, whatever the dimension, zero-stability of these methods is equivalent to zero-stability of the basic method (20) which, in turn, may be estabilished by looking at the location of the parasitic root α. Thus method (A_1^e, B_1^e) is zero-stable if $|\alpha| < 1$ and zero-unstable otherwise, while a reverse condition holds for the method (A_2^e, B_2^e). We used for α the values 0.5 and 2.5 and applied the two methods to the problem

$$\begin{cases} y'(t) = sin(t)y(t) & t \in [0, 2\pi] \\ y(0) = 1 \end{cases} . \tag{22}$$

In Figures (1) and (2) the errors generated by the two methods performed with dimensions $N = 2, 4, 6, 8$ are compared for several values of the stepsize h. We observe that for all examined methods the matrices A_1^e and A_2^e are stable in the sense of the definition of stability given for GLMs and indeed the sequence of the errors produced by each method goes to zero when h decreases. On the other hand, looking at these figures and Table (1) the sensitivity of the conditioning of the discrete problem with respect to the conditioning of A^{-1} is clearly visible: this sensitivity is minimized if one works with zero-stable block BVMs.

Figure 1: Conditioning of (A_1^e, B_1^e) used with $\alpha = 2.5$ (left) and $\alpha = 0.5$ (right)

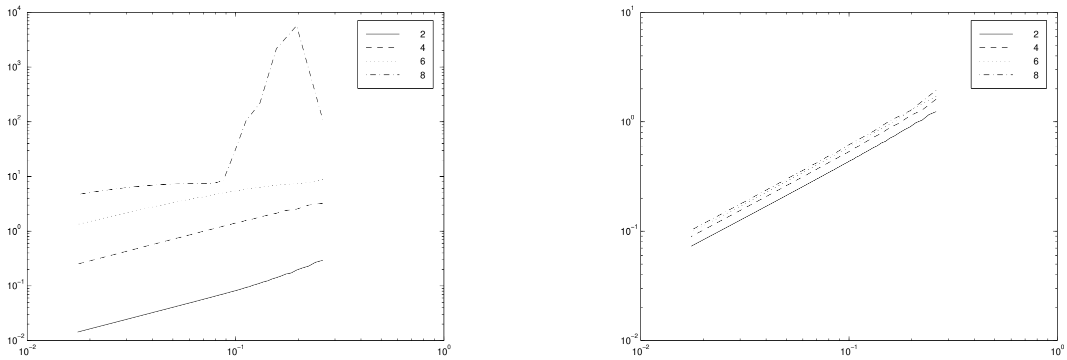


Figure 2: Conditioning of (A_2^e, B_2^e) used with $\alpha = 2.5$ (left) and $\alpha = 0.5$ (right)

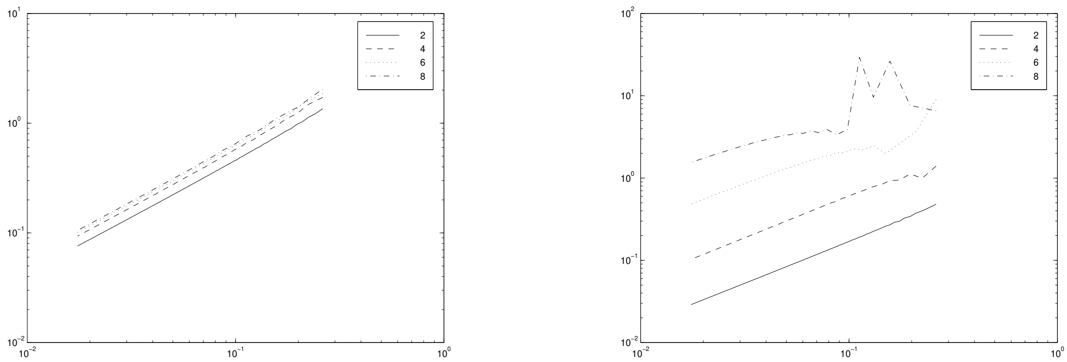


Table 1: Conditioning of the methods (A_1^e, B_1^e) (left) and (A_2^e, B_2^e) (right).

α	N	$S \simeq \frac{\|\|e^{(n)}\|\|_\infty}{h_n}$	$\|\|A^{-1}\|\|_\infty$
2.5	2	0.8	1.0
	4	14.3	5.9
	6	76.1	42.8
	8	268.4	275.7
0.5	2	4.1	1.2
	4	5.1	2.4
	6	5.5	3.6
	8	5.8	4.9

α	N	$S \simeq \frac{\|\|e^{(n)}\|\|_\infty}{h_n}$	$\|\|A^{-1}\|\|_\infty$
2.5	2	4.3	1.5
	4	5.3	2.2
	6	5.8	2.8
	8	6.0	3.5
0.5	2	1.6	4.5
	4	5.9	36.9
	6	27.7	202.8
	8	88.4	983.4

3 Linear stability of block BVMs.

According to the scheme presented in the previous section, when a family of block-BVMs is used to solve the test problem

$$\begin{cases} y'(t) = \lambda y, & t \in [0, +\infty[\\ y(0) = y_0, \end{cases}, \tag{23}$$

it provides a sequence of approximations $\mathbf{y}^{(s)} = [y_1^{(s)}, \dots, y_{N_s}^{(s)}]^T$, $s = 1, 2, \dots$, each of which is the solution of the linear system

$$(A_s - h_s \lambda B_s)\, \mathbf{y}^{(s)} = (-\mathbf{a}_0^{(s)} + h_s \lambda \mathbf{b}_0) y_0^{(s)}. \tag{24}$$

In the concrete cases this family contains a finite number of different methods. Under this assumption we gather all different pairs and form the sequence $\Gamma \equiv (A_i^e, B_i^e)_{i \in I}$, where I is a finite set of indices satisfying

(i) $\forall i, j \in I,\ i \neq j\ :\ \ (A_i^e, B_i^e) \neq (A_j^e, B_j^e)$;

(ii) for any integer s : $(A_s^e, B_s^e) \in \Gamma$.

Here we are interested in investigating the conditions to be satisfied by each pair (A_s^e, B_s^e) in order to get a bounded sequence $(\mathbf{y}^{(s)})_{s=1}^{\infty}$. Initially we restrict our study to the case of a constant family of block-BVMs ($card(I) = 1$), performed with constant stepsize h, since the behavior of this simplified model contains most of the features of the more complex one introduced above. Hence we suppose to proceed in the integration of (23) by means of a single pair (A^e, B^e) or equivalently by its associated sequence (8): system (24) can be recast as:

$$(A - qB)\, \mathbf{y}^{(s)} = (-\mathbf{a}_0 + q\mathbf{b}_0) y_0^{(s)}, \tag{25}$$

where $q = h\lambda$. If the matrix P is defined as

$$P = (A - qB)^{-1}[\mathbf{0}, \dots, \mathbf{0}, -\mathbf{a}_0 + q\mathbf{b}_0] = [\mathbf{0}, \dots, \mathbf{0}, (A - qB)^{-1}(-\mathbf{a}_0 + q\mathbf{b}_0)],$$

the link between $\mathbf{y}^{(s)}$ and $\mathbf{y}^{(s-1)}$ is then expressed by the difference equation

$$\mathbf{y}^{(s)} = P\mathbf{y}^{(s-1)}.$$

Consequently the null vector of $\mathbb{R}^N$ is asymptotically stable iff the spectral radius of P is less than one and this, as easily seen from the definition of P, means that the linear system

$$(A - qB)\, \mathbf{x} = (-\mathbf{a}_0 + q\mathbf{b}_0), \tag{26}$$

admits a solution $\mathbf{x}(q) = [x_1(q), \dots, x_N(q)]^T$ satisfying $|x_N(q)| < 1$. These considerations allow one to identify the stability region of a block-BVM as the subset of the complex plane:

$$\mathcal{S} = \left\{ q \in \mathbb{C} : \mathbf{x}(q) = (A - qB)^{-1}(-\mathbf{a}_0 + q\mathbf{b}_0) \ \text{ satisfies } \ |x_N(q)| < 1 \right\}.$$

Concerning the way of plotting the boundary of a stability domain, any standard approach used for Runge-Kutta schemes may be adopted (see [10, 12, 18]). In the next section we report graphs of the regions $\mathcal{S}$ corresponding to some classes of block-BVMs. We made use of an alternative technique which avoids the preliminary determination of the stability function and consists in recasting (26) in the form

$$(A^e - qB^e)\begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix} = \mathbf{0}, \tag{27}$$

then setting $x_N = e^{i\theta}$ (i is the imaginary unit) and finally retaining q as an unknown. In this way (27) becomes a nonlinear system containing N equations and N unknowns, namely $x_1, \ldots, x_{N-1}$ and q. If we introduce the function

$$G(\mathbf{u}, \theta) = (A^e - u_N B^e)\begin{pmatrix} 1 \\ u_1 \\ \vdots \\ u_{N-1} \\ e^{i\theta} \end{pmatrix},$$

and make the assumption that $u_i = x_i,\ i = 1, \ldots, N-1$ and $u_N = q$, the problem becomes that of finding the zeros of a parametric system defined in the finite dimensional space $\mathbb{R}^N$. The solution set of G is defined as

$$\mathcal{F} = \left\{(\mathbf{u}, \theta) \in \mathbb{R}^N \times \mathbb{R}\ :\ G(\mathbf{u}, \theta) = 0\right\}$$

and contains the boundary of $\mathcal{S}$.

Constructive methods for solving such kinds of systems generate a set of paths

$$\Gamma_i = \{(\mathbf{u}_i(s), \theta_i(s))\ :\ G(\mathbf{u}_i(s), \theta_i(s)) = 0, s \in I_i\}$$

where I_i is a real interval. The solution set is then the union of a finite number of connected components. Concerning the theory and the numerical treatment of these problems see for example [9, 16, 19, 20]. For our purpose we have constructed a Matlab code based on the algorithms presented in [16]. A method for the determination of the set $\mathcal{F}$ must include the possibility of handling singular points such as limit and bifurcation points in correspondence of which the implicit function theorem may not be applied and the standard methods (such as those based on Newton methods) fail during natural parameter continuation. As usual, after the paths Γ_i have been determined, one should check the location of the poles of the stability function which in our case are the zeros of $det(A - qB)$ and then apply the maximum principle to obtain the set $\mathcal{S}$.

In the more general case represented by equations (24), the non autonomous system

$$\mathbf{y}^{(s)} = P(s)\mathbf{y}^{(s-1)} \tag{28}$$

must be solved with

$$P(s) = (A_s - q_s B_s)^{-1}[\mathbf{0}, \ldots, \mathbf{0}, -\mathbf{a}_0^{(s)} + q_s \mathbf{b}_0^{(s)}],$$

and $q_s = h_s\lambda$. The null vector is still an equilibrium point for (28) and its stability behavior may be studied by means of suitable criteria (see for example [17]) one of which is reported as follows.

Lemma 3.1 *Consider the sequence of matrices*

$$\Phi(n) = \prod_{s=1}^{n} P(s).$$

Then the solution $\mathbf{y} = 0$ *of (28) is stable iff there exists* $M > 0$ *such that* $||\Phi(n)|| < M \ \forall n \geq 1$*, and is asymptotically stable iff* $||\Phi(n)|| < a\eta^n \ \forall n \geq 1$*, where* η *and* a *are positive constants independent of* n*, with* $\eta < 1$.

In our case, due to the particular structure of $P(s)$, if $x_{N_s}^{(s)}$ is the nontrivial diagonal entry of $P(s)$, we have

$$\Phi(n) = \left(\prod_{s=1}^{n-1} x_{N_s}^{(s)}\right) P(n). \tag{29}$$

Now we suppose that a positive integer s^* exists such that q_s belongs to $\mathcal{S}_s$ for any $s \geq s^*$ and we observe that the constants

$$M_1 = \max_{s\in I}\left(\max_{q\in\bar{\mathcal{S}}_s} ||(A_s - qB_s)^{-1}||\right)$$

and

$$M_2 = \max_{s\in I}\left(\max_{q\in\bar{\mathcal{S}}_s} || - \mathbf{a}_0^{(s)} + q\mathbf{b}_0^{(s)}||\right)$$

are well defined due to the continuity on $\bar{\mathcal{S}}_s$, the closure of $\mathcal{S}_s$, of the functions $f_s(q) = ||(A_s - qB_s)^{-1}||$ and $g_s(q) = || - \mathbf{a}_0^{(s)} + q\mathbf{b}_0^{(s)}||$ (here $||\cdot||$ denotes $||\cdot||_1$, $||\cdot||_2$ or $||\cdot||_\infty$). From (29) we get

$$\begin{aligned} ||\Phi(n)|| &\leq \prod_{s=1}^{n-1} |x_{N_s}^{(s)}| \, ||P(n)|| \leq \prod_{s=1}^{n-1} |x_{N_s}^{(s)}| \, ||(A_n - q_nB_n)^{-1}|| \, || - \mathbf{a}_0^{(s)} + q_n\mathbf{b}_0^{(s)}|| \\ &\leq M \prod_{s=1}^{n-1} |x_{N_s}^{(s)}|. \end{aligned}$$

where $M = M_1M_2$. From this last inequality a sufficient stability result may be easily deduced.

THEOREM 3.2 *Suppose the test problem (23) is solved by a finite family of block-BVMs* $(A_i^e, B_i^e)_{i\in I}$ *each of which has a nonempty region of A-stability* S_i*. For any* $i \in I$ *consider a set* $\widetilde{\mathcal{S}}_i \subset \mathcal{S}_i$ *such that* $\widetilde{\mathcal{S}}_i$ *and* $\mathcal{S}_i$ *have disjoint boundaries. Then the following statements hold true*

(i) the numerical solution $\mathbf{y}^{(s)}$ *is bounded if a positive integer* s^* *exists such that* $q_s \in \mathcal{S}_s \ \forall s \geq s^*$*;*

(ii) the numerical solution $\mathbf{y}^{(s)}$ *converges to zero if a positive integer* s^* *exists such that* $q_s \in \widetilde{\mathcal{S}}_s \ \forall s \geq s^*$.

Concluding this section, we observe that, applying Cramer's rule, the stability region $\mathcal{S}$ may be also well described as follows:

$$\mathcal{S} = \left\{ q \in \mathbb{C} : |det(A - qB)| < |det(A^{(1)} - qB^{(1)})| \right\}. \tag{30}$$

where $A^{(1)},\ B^{(1)}$ are $N \times N$ matrices defined, respectively, by the partitions

$$A^e = [A^{(1)}, \mathbf{a}_N], \qquad B^e = [B^{(1)}, \mathbf{b}_N].$$

4 Generalized Adams Methods

We now present a particular family of block-BVMs, namely the Generalized Adams Methods (GAMs), that were used to construct a code for the solution of initial value problems [15]. Their description is given by listing the basic, initial and final methods. As we shall see some of the GAMs belong to the so-called symmetric schemes which were conveniently defined in [6].

Definition 4.1 *Consider the permutation matrices*

$$P_1 = \begin{pmatrix} & & 1 \\ & \cdot & \\ 1 & & \end{pmatrix}_{N\times N}, \qquad P_2 = \begin{pmatrix} & 1 \\ P_1 & \end{pmatrix}_{(N+1)\times(N+1)}.$$

A block-BVM (A^e, B^e) is symmetric if it verifies

$$\begin{array}{rl} P_1 A^e P_2 = & -A^e, \\ P_1 B^e P_2 = & B^e. \end{array}$$

An important property satisfied by symmetric schemes is that the boundary of their stability regions contains the imaginary axis. As a consequence such methods are candidates to belong to the class of perfectly A-stable methods for which the stability region coincides with the left half complex plane. To see this, we start with a lemma.

Lemma 4.2 *For a symmetric block-BVM (A^e, B^e), the following properties hold true:*

(i) $P_1 B P_1 = B^{(1)}$;

(ii) $P_1 A P_1 = -A^{(1)}$.

Proof. By definition

$$[B^{(1)}, \mathbf{b}_N] = P_1[\mathbf{b}_0, B]P_2 = [P_1 B P_1, P_1 \mathbf{b}_0]$$

and this implies (i). Property (ii) is proved by an analogous procedure. □

Theorem 4.3 *For a symmetric block-BVM (A^e, B^e), the boundary of the region of A-stability contains the imaginary axis.*

Proof. Using the results stated in Lemma 4.2, we have for any $\alpha \in \mathbb{R}$,

$$|det(A - \alpha i B)| = |det(A + \alpha i B)| = |det(-A^{(1)} + \alpha i B^{(1)})| = |det(A^{(1)} - \alpha i B^{(1)})|,$$

and this, on the basis of formula (30) proves that $q = \alpha i$ belongs to the boundary of $\mathcal{S}$. □

The k-step GAM is identified by the k multistep methods

$$\rho_i(\xi) = \xi^i(1-\xi^{-1}), \qquad \sigma_i(\xi) = \xi^i \sum_{j=-i}^{k-i} \beta_{ij}\xi^j, \quad i = 1,\dots,k \tag{31}$$

where the coefficients β_{ij} are uniquely determined by imposing that the method has maximum order (i.e. $k+1$). Defining the index ν as

$$\nu = \begin{cases} (k+1)/2 & \text{for odd k,} \\ k/2 & \text{for even k,} \end{cases}$$

the initial, basic and final methods are represented by formulae (31) respectively for $i = 1,\dots,\nu-1$, $i = \nu$ and $i = \nu+1,\dots,k$. Due to the expression of ρ_i (that justifies the name of these methods) we see that the matrix A^e is bidiagonal and Toeplitz with 1 on the main diagonal and -1 on the first lower diagonal. In table (2) we report the coefficients β_{ij} for the GAMs up to the order 9. Once an order p has been fixed, the initial, basic and final methods are labelled and ordered in the second column, while the values of β_{ij} are obtained by multiplying the integers appearing in the fourth column of the table by the normalizing factor (nf) corresponding to the chosen method. To make the table more clear, we have marked in bold the coefficient that will appear in the diagonal position when constructing the matrix B and in fact all β_{ij} in the table corresponding to a given method form the matrix B^e of minimum dimension.

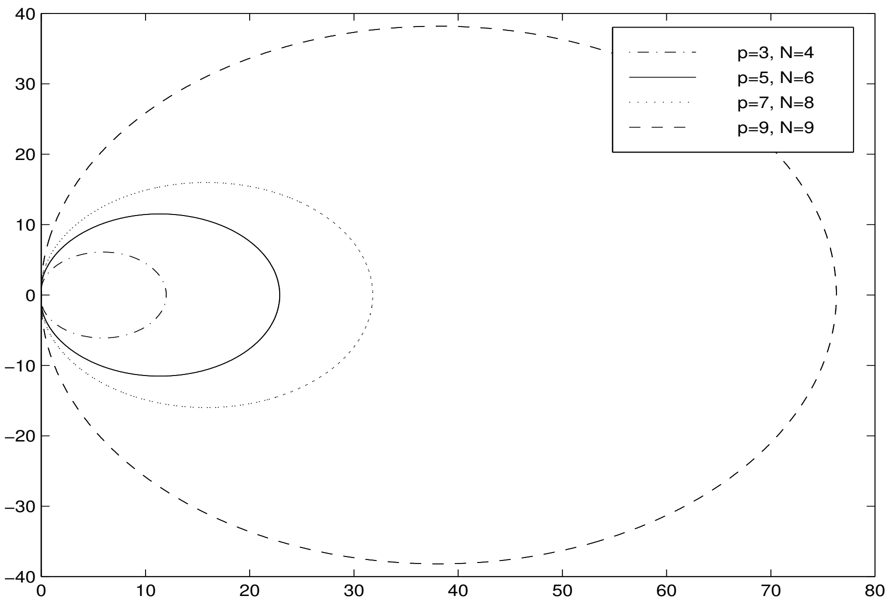
Table 2: GAMs of order up to 9.

p		nf	coefficients β_{ij}								
2	b	1/2	1	**1**							
3	b	1/12	5	**8**	−1						
	f_1		−1	8	**5**						
4	i_1	1/24	9	**19**	−5	1					
	b		−1	13	**13**	−1					
	f_1		1	−5	19	**9**					
5	i_1	1/720	251	**646**	−264	106	−19				
	b		−19	346	**456**	−74	11				
	f_1		11	−74	456	**346**	−19				
	f_2		−19	106	−264	646	**251**				
6	i_1	1/1440	475	**1427**	−798	482	−173	27			
	i_2		−27	637	**1022**	−258	77	−11			
	b		11	−93	802	**802**	−93	11			
	f_1		−11	77	−258	1022	**637**	−27			
	f_2		27	−173	482	−798	1427	**475**			
7	i_1	1/60480	19087	**65112**	−46461	37504	−20211	6312	−863		
	i_2		−863	25128	**46989**	−16256	7299	−2088	271		
	b		271	−2760	30819	**37504**	−6771	1608	−191		
	f_1		−191	1608	−6771	37504	**30819**	−2760	271		
	f_2		271	−2088	7299	−16256	46989	**25128**	−863		
	f_3		−863	6312	−20211	37504	−46461	65112	**19087**		
8	i_1	1/120960	36799	**139849**	−121797	123133	−88547	41499	−11351	1375	
	i_2		−1375	47799	**101349**	−44797	26883	−11547	2999	−351	
	i_3		351	−4183	57627	**81693**	−20227	7227	−1719	191	
	b		−191	1879	−9531	68323	**68323**	−9531	1879	−191	
	f_1		191	−1719	7227	−20227	81693	**57627**	−4183	351	
	f_2		−351	2999	−11547	26883	−44797	101349	**47799**	−1375	
	f_3		1375	−11351	41499	−88547	123133	−121797	139849	**36799**	
9	i_1	1/3628800	1070017	**4467094**	−4604594	5595358	−5033120	3146338	−1291214	312874	−33953
	i_2		−1375	1375594	**3244786**	−1752542	1317280	−755042	294286	−68906	7297
	i_3		7297	−99626	1638286	**2631838**	−833120	397858	−142094	31594	−3233
	b		−3233	36394	−216014	1909858	**2224480**	−425762	126286	−25706	2497
	f_1		2497	−25706	126286	126286	2224480	**1909858**	−216014	36394	−3233
	f_2		−3233	31594	−142094	397858	−833120	2631838	**1638286**	−99626	7297
	f_3		7297	−68906	294286	−755042	1317280	1317280	3244786	**1375594**	−33953
	f_4		−33953	312874	−1291214	3146338	−5033120	5595358	−4604594	4467094	**1070017**

The GAMs corresponding to an even order are called Extended Trapezoidal Rules (ETRs) because for $p = 2$ one gets the classic trapezoidal formula and, like this method, the ETRs are symmetric schemes. It may be verified that they are perfectly A-stable up to order 8. In the scheme of order 10 two roots of $det(A - qB)$ are located in $\mathbb{C}^-$. The odd order schemes are symmetric only when the lowest dimension N is used; otherwise they fail symmetry because of the unsymmetry of the coefficients of the basic method. In figure (3) the stability regions of the GAMs with order $p = 3,5,7,9$, and dimensions respectively $N = 4,6,8,9$ are shown (stability occurs outside the closed curves). These particular choices of the dimension allow us to perform a procedure to control the error and change the order and the stepsize of the scheme (see [14]).

In figure (4) the third order formula is used to put in relation the stability region of the basic method $\mathcal{S}_b$ with those ($\mathcal{S}_N$) corresponding to the dimensions $N = 3,4,5,20$. It may be ar-

Figure 3: Stability regions of some odd order GAMs



gued (but this remains an open question) that $\mathcal{S}_\infty = \mathcal{S}_b$. The important matter is that even for small N the boundary locus of the block-BVM is very similar to that of the basic method $(\{q = \rho_b(e^{i\theta})/\sigma_b(e^{i\theta}), \theta \in [0, 2\pi]\})$ and this in turn establishes a link between the basic method and the block-BVM on the basis of which some structural features of the former are transferred to the latter. For example, as the basic formula, all the considered odd GAMs are L-stable, a desirable property when dealing with dissipative systems.

5 Conclusions

We have seen that the block-BVMs are interesting for a number of reasons, such as the simplicity of their formulation, the expression of the local truncation error, and the possibility of inheriting qualitative properties satisfied by the basic methods. Other properties that concern their implementation, such as the easiness in changing the order, are analyzed in [14]. Although they may be formally classified as one-step implicit methods, there is something novel about a block-BVM formula, namely the possibility of changing the dimension N of the block. The importance of that is also clear if one wants to solve the nonlinear system (3) in parallel: if the architecture consists of p processors, one could fix $\tilde{N} = p$ hence adapting the same code on different machines and so optimizing the potentiality offered by each hardware.

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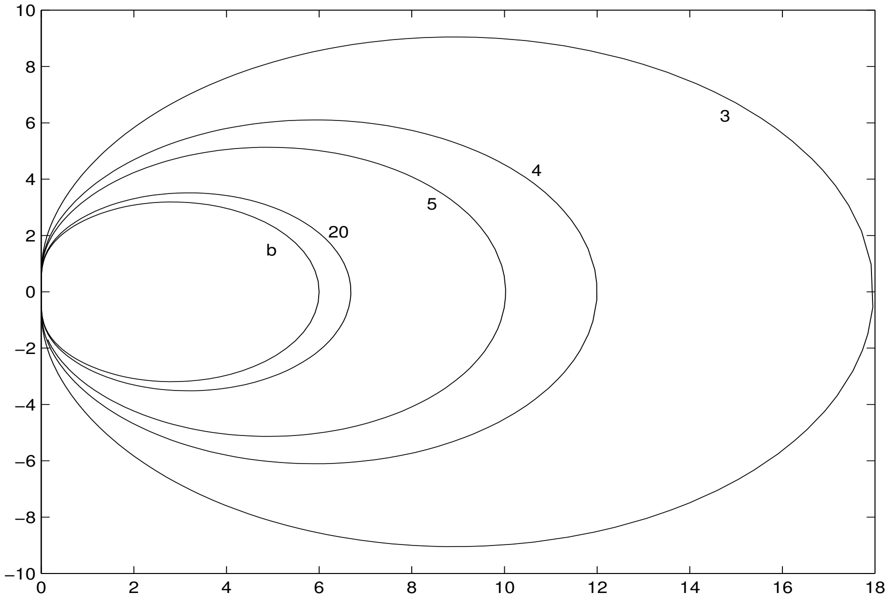
Figure 4: Stability regions of the GAM of order 3 for several dimensions N



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